

Lagrange–Fedosov Nonholonomic Manifolds

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Abstract

We outline an unified approach to geometrization of Lagrange mechanics, Finsler geometry and geometric methods of constructing exact solutions with generic off-diagonal terms and nonholonomic variables in gravity theories. Such geometries with induced almost symplectic structure are modelled on nonholonomic manifolds provided with nonintegrable distributions defining nonlinear connections. We introduce the concept of Lagrange–Fedosov spaces and Fedosov nonholonomic manifolds provided with almost symplectic connection adapted to the nonlinear connection structure. We investigate the main properties of generalized Fedosov nonholonomic manifolds and analyze exact solutions defining almost symplectic Einstein spaces.

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I Introduction

The geometry of Fedosov manifolds is a natural generalization of Kähler geometry defining a procedure of canonical deformation quantization^{1–5}. By definition, a Fedosov manifold is given by a triple (M, θ, Γ) where M is a \mathcal{C}^∞ -manifold enabled with symplectic structure θ (a non-degenerated closed exterior 2-form) and a symplectic connection structure Γ (i.e. a torsionless linear connection parallelizing the symplectic form). If a Lagrange fundamental function $L : (x, y) \in TM \rightarrow \mathbb{R}$ is defined on M ^a, there is a natural almost complex structure adapted to the canonical nonlinear connection (in brief, N-connection) induced by $L(x, y)$ ^{6–8}. Nonlinear connections can be also naturally related to generic off-diagonal metrics and nonholonomic moving frames in (super) gravity and string theories^{9–13}. So, if we want to apply the methods of symplectic geometry (and possible generalizations for Poisson manifolds¹⁴) to various type of Lagrange–Hamilton, and related Finsler–Cartan spaces, we have to consider spaces enabled with N-connection structure.

In this work, we study the geometry of almost symplectic connections (in general, they are not torsion-free but can be symmetrized) which are distinguished by a N-connection structure and preserve an almost symplectic form, for instance, induced by a regular Lagrangian or off-diagonal metric structure. This is related to almost symplectic manifolds (see, for instance^{15–19}) but, in our case, the manifolds are nonholonomic ones.

We shall define and analyze the curvature tensor for such almost symplectic connections and related Einstein equations with nonholonomic variables. For nonholonomic manifolds, i.e., manifolds with nonintegrable distributions (in our case, with a such distribution defined by a N-connection), this is not a trivial task. The problem together with a proposal when the Riemann tensor is interpreted as a modification of the Spencer cohomology and related to solutions of partial differential equations, as well to superspaces, are analyzed in^{20,21}.

The geometry of nonholonomic manifolds has a long time historical perspective: For instance, in the review²² it is stated that it is probably impossible to construct an analog of the Riemannian tensor for the general nonholonomic manifold. In two more recent reviews^{23,24}, it is emphasized that in the past there were proposed well defined Riemannian tensors for a number of spaces provided with nonholonomic distributions, like Finsler and Lagrange spaces and various type of theirs higher order generalizations, i.e., for non-

^aFor simplicity, in this work we shall consider only regular Lagrangians; (x, y) denote a set of local coordinates on the tangent bundle TM with $x \in M$.

holonomic manifolds possessing corresponding N-connection structures. As some examples of former such investigations, we cite the works^{25–29}.

Essentially, the Fedosov type nonholonomic geometry to be elaborated in this work is based on the notion of N-connection and considers a Whitney-like splitting of the tangent bundle to a manifold into horizontal and vertical subspaces (see discussion and a bibliography for recent developments and applications in^{30–32}). Here we emphasize that the geometrical aspects of the N-connection formalism has been studied since the first papers of E. Cartan³³ and A. Kawaguchi^{34–36} (who used it in component form for Finsler geometry), then one should be mentioned the so called Ehresmann connection³⁷ and the work of W. Barthel³⁸ where the global definition of N-connection was given. The monographs^{6–8} consider the N-connection formalism elaborated and applied to the geometry of generalized Finsler–Lagrange and Cartan–Hamilton spaces, see also the approaches^{39–42}.

The works related to nonholonomic geometry and N-connections have appeared many times in a rather dispersive way when different school of authors from geometry, mechanics and physics have worked many times not having relation with another. We outline some recent results with explicit applications in modern mathematical physics and particle and string theories: N-connection structures were modelled on Clifford and spinor bundles^{43,44}, on superbundles and in some directions of (super) string theory^{45,46}, as well in noncommutative geometry and gravity⁴⁷. The idea to apply the N-connections formalism as a new geometric method of constructing exact solutions in gravity theories was suggested in^{9,10} and developed in a number of works, see for instance^{11–13}).

We begin in Section II with an introduction into the N-connection geometry for arbitrary manifolds with tangent bundles admitting splitting into conventional horizontal and vertical subspaces. We illustrate how regular Lagrangians induce natural semispray, N-connection, metric and almost complex structures on tangent bundles and discuss the relation between Lagrange and Finsler geometry and theirs generalizations. Then we prove that N-connection structures and corresponding almost complex geometries may be modelled by generic off-diagonal metrics and nonholonomic frames in gravity theories.

Section III is devoted to the theory of linear connections on N-anholonomic manifolds (i.e., on manifolds with nonholonomic structure defined by N-connections). We demonstrate how the linear connections may be adapted to the N-connection splitting of the manifolds and analyze the conditions when such distinguished connections may be naturally related to almost complex structures. This has great *philosophical* interest, because several authors have defined different notions of general connections, looking for associated

parallel transport and covariant differential operator satisfying, if possible, the properties of those of a linear connection (e.g., Ehresmann connections on bundles, non-homogeneous connections of Grifone⁴⁸, quasi- and pseudo-connections (see the survey⁴⁹, etc.), but it always implies to lose properties or to demand more assumptions than in the case of a N-connection^b. In the present paper we shall see that one can define a canonical linear connection adapted to a given N-connection. This shall avoid us extra constructions and additional restrictions.

In Section IV, we define the Fedosov N-anholonomic and Lagrange-Fedosov manifolds as certain generalizations of the Fedosov spaces to nonholonomic configurations. We construct in explicit form the curvature tensor of such spaces and define the Einstein equations for N-adapted linear connection and metric structures.

In Section V, we analyze the main conditions when vacuum gravitational configurations with N-anholonomic structures can be defined as exact solutions of the Einstein equations. We prove that for a very general five dimensional ansatz for metric coefficients depending on two, three and four variables the system of field equations is completely integrable. We illustrate that the method can be reduced to the case of four dimensional spaces which gives us the possibility to generate conformal almost complex gravitational metrics.

We shall use both physical and mathematical languages and both coordinate and intrinsic notations, when possible.

II Nonlinear Connections and Fedosov Spaces

In this section, we recall some results on nonlinear connections and almost symplectic structures, which in certain particular cases, are induced by regular Lagrangians, Finsler fundamental functions or by generic off-diagonal metrics in gravity theories. From now on, all the manifolds (in general, non-holonomic ones)^c and geometric objects are supposed to be \mathcal{C}^∞ .

^bFor example, non-homogeneous connections of Grifone define a covariant derivative $D_X Y$, which in general does not define a vector field on the manifold (see p. 302 of the reference⁴⁸) and which does not satisfy $D_X(fY) = fD_X Y + (Xf)Y$ (see p. 305 of the same reference). In the case of nonlinear connections used in the book of Yano and Ishihara⁵⁰ p. 209 it is assumed that horizontal distributions are invariant under dilatations (see also⁵¹), etc.

^cIn literature, it is also used the equivalent term: anholonomic.

A Nonlinear connection geometry

Let \mathbf{V} be a $(n+m)$ -dimensional manifold. It is supposed that in any point $u \in \mathbf{V}$ there is a local splitting $\mathbf{V}_u = M_u \oplus V_u$, where M is a n -dimensional subspace and V is a m -dimensional subspace^d. We shall split the local coordinates (in general, abstract ones both for holonomic and nonholonomic variables) in the form $u = (x, y)$, or $u^\alpha = (x^i, y^a)$, where $i, j, k, \dots = 1, 2, \dots, n$ and $a, b, c, \dots = n+1, n+2, \dots, n+m$. We denote by $\pi^\top : T\mathbf{V} \rightarrow TM$ the differential of a map $\pi : V^{n+m} \rightarrow V^n$ defined by fiber preserving morphisms of the tangent bundles $T\mathbf{V}$ and TM . The kernel of π^\top is just the vertical subspace $v\mathbf{V}$ with a related inclusion mapping $i : v\mathbf{V} \rightarrow T\mathbf{V}$.

Definition 1 *A nonlinear connection (N-connection) \mathbf{N} on a manifold \mathbf{V} is defined by the splitting on the left of an exact sequence*

$$0 \rightarrow v\mathbf{V} \xrightarrow{i} T\mathbf{V} \rightarrow T\mathbf{V}/v\mathbf{V} \rightarrow 0,$$

i. e. by a morphism of submanifolds $\mathbf{N} : T\mathbf{V} \rightarrow v\mathbf{V}$ such that $\mathbf{N} \circ i$ is the unity in $v\mathbf{V}$.

In an equivalent form, we can say that a N-connection is defined by a splitting to subspaces with a Whitney sum of conventional horizontal (h) subspace, $(h\mathbf{V})$, and vertical (v) subspace, $(v\mathbf{V})$,

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \quad (1)$$

where $h\mathbf{V}$ is isomorphic to M . Moreover, one can say that a N-connection is defined by a tensor field of type (1,1) $\mathbf{P} = \mathbf{H} - \mathbf{N}$, where \mathbf{H} (resp. \mathbf{N}) denotes the projection over the horizontal (resp. vertical) subspace. Observe that $\mathbf{P} \circ \mathbf{P} = \mathbf{I}$, i.e., \mathbf{P} is an almost product structure, horizontal (resp. vertical) subspace being the eigenspace associated to the eigenvalue +1 (resp. -1).

Locally, a N-connection is defined by its coefficients $N_i^a(u)$,

$$\mathbf{N} = N_i^a(u) dx^i \otimes \frac{\partial}{\partial y^a}.$$

The well known class of linear connections consists on a particular subclass with the coefficients being linear on y^a , i.e., $N_i^a(u) = \Gamma_{bj}^a(x) y^b$.

Any N-connection $\mathbf{N} = N_i^a(u)$ may be characterized by an associated frame (vielbein) structure $\mathbf{e}_\nu = (e_i, e_a)$, where

$$e_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a} \text{ and } e_a = \frac{\partial}{\partial y^a}, \quad (2)$$

^dOne has this local decomposition when $\mathbf{V} \rightarrow M$ is a surjective submersion. A particular case is that of a fibre bundle, but we can obtain the results in the general case.

and the dual frame (coframe) structure $\vartheta^\mu = (\vartheta^i, \vartheta^a)$, where

$$\vartheta^i = dx^i \text{ and } \vartheta^a = dy^a + N_i^a(u)dx^i. \quad (3)$$

These vielbeins are called N-adapted frames. In order to preserve a relation with the previous denotations^{8–11,43,44,46}, we note that $\mathbf{e}_\nu = (e_i, e_a)$ and $\vartheta^\mu = (\vartheta^i, \vartheta^a)$ are, respectively, the former $\delta_\nu = \delta/\partial u^\nu = (\delta_i, \partial_a)$ and $\delta^\mu = \delta u^\mu = (d^i, \delta^a)$ which emphasize that operators (2) and (3) define, correspondingly, certain “N-elongated” partial derivatives and differentials which are more convenient for calculations on such nonholonomic manifolds.

Any N-connection also defines a N-connection curvature

$$\Omega = \frac{1}{2}\Omega_{ij}^a d^i \wedge d^j \otimes \partial_a,$$

with N-connection curvature coefficients

$$\Omega_{ij}^a = \delta_{[j} N_{i]}^a = \delta_j N_i^a - \delta_i N_j^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \quad (4)$$

The vielbeins (3) satisfy the nonholonomy (equivalently, anholonomy) relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma \quad (5)$$

with (antisymmetric) nontrivial anholonomy coefficients $W_{ia}^b = \partial_a N_i^b$ and $W_{ji}^a = \Omega_{ij}^a$.

Definition 2 *A manifold \mathbf{V} is called N-anholonomic if on the tangent space $T\mathbf{V}$ it is defined a local (nonintegrable) distribution (1), i.e., $T\mathbf{V}$ is enabled with a N-connection and related nonholonomic vielbein structure (5).*

We note that in this work we use boldfaced symbols for the spaces and geometric objects provided/adapted to a N-connection structure. For instance, a vector field $\mathbf{X} \in T\mathbf{V}$ is expressed $\mathbf{X} = (X, {}^\vee X)$, or $\mathbf{X} = X^\alpha \mathbf{e}_\alpha = X^i e_i + X^a e_a$, where $X = X^i e_i$ and ${}^\vee X = X^a e_a$ state, respectively, the irreducible (adapted to the N-connection structure) horizontal (h) and vertical (v) components of the vector (which following refereces^{6,7} is called a distinguished vectors, in brief, d-vector). In a similar fashion, the geometric objects on \mathbf{V} like tensors, spinors, connections, ... are called respectively d-tensors, d-spinors, d-connections if they are adapted to the N-connection splitting.

In the next two subsections we show how certain type of N-connection geometries can be naturally derived from Lagrange-Finsler geometry and in gravity theories.

B N-connections and Lagrangians

We outline the main results on N-connections and almost symplectic structures induced by regular Lagrangians^{6–8}. In this case the N-anholonomic manifold \mathbf{V} is to be modelled on the tangent bundle (TM, π, M) , where M is a n -dimensional base manifold, π is a surjective projection and TM is the total space. One denotes by $\widetilde{TM} = TM \setminus \{0\}$ where $\{0\}$ means the null section of map π .

A differentiable Lagrangian $L(x, y)$, i. e. a fundamental Lagrange function, is defined by a map $L : (x, y) \in TM \rightarrow L(x, y) \in \mathbb{R}$ of class \mathcal{C}^∞ on \widetilde{TM} and continuous on the null section $0 : M \rightarrow TM$ of π . A regular Lagrangian is with nondegenerated Hessian,

$${}^{(L)}g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \quad (6)$$

when $\text{rank } |g_{ij}| = n$ on \widetilde{TM} .

Definition 3 A Lagrange space is a pair $L^n = [M, L(x, y)]$ with ${}^{(L)}g_{ij}(x, y)$ being of constant signature over \widetilde{TM} .

The notion of Lagrange space was introduced by J. Kern⁵² and elaborated in details by the R. Miron's school on Finsler and Lagrange geometry, see references^{6,7}, as a natural extension of Finsler geometry^{33,53–59} (see also references^{45,46}, on Lagrange-Finsler supergeometry).

By straightforward calculations, there where proved the results:

1. The Euler-Lagrange equations

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0$$

where $y^i = \frac{dx^i}{d\tau}$ for $x^i(\tau)$ depending on parameter τ , are equivalent to the “nonlinear” geodesic equations

$$\frac{d^2 x^i}{d\tau^2} + 2G^i(x^k, \frac{dx^j}{d\tau}) = 0$$

defining paths of a canonical semispray

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$$

where

$$2G^i(x, y) = \frac{1}{2} {}^{(L)}g^{ij} \left(\frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right)$$

with ${}^{(L)}g^{ij}$ being inverse to (6).

2. There exists on \widetilde{TM} a canonical N-connection

$${}^{(L)}N_j^i = \frac{\partial G^i(x, y)}{\partial y^j} \quad (7)$$

defined by the fundamental Lagrange function $L(x, y)$, which prescribes nonholonomic frame structures of type (2) and (3), ${}^{(L)}\mathbf{e}_\nu = (e_i, {}^\vee e_k)$ and ${}^{(L)}\vartheta^\mu = (\vartheta^i, {}^\vee \vartheta^k)$.^e

3. The canonical N-connection (7), defining ${}^\vee e_i$, induces naturally an almost complex structure $\mathbf{F} : \chi(\widetilde{TM}) \rightarrow \chi(\widetilde{TM})$, where $\chi(\widetilde{TM})$ denotes the module of vector fields on \widetilde{TM} ,

$$\mathbf{F}(e_i) = {}^\vee e_i \text{ and } \mathbf{F}({}^\vee e_i) = -e_i,$$

when

$$\mathbf{F} = {}^\vee e_i \otimes \vartheta^i - e_i \otimes {}^\vee \vartheta^i \quad (8)$$

satisfies the condition $\mathbf{F} \rfloor \mathbf{F} = -\mathbf{I}$, i. e. $F^\alpha{}_\beta F^\beta{}_\gamma = -\delta_\gamma^\alpha$, where δ_γ^α is the Kronecker symbol and “ \rfloor ” denotes the interior product.

4. On \widetilde{TM} , there is a canonical metric structure

$${}^{(L)}\mathbf{g} = {}^{(L)}g_{ij}(x, y) \vartheta^i \otimes \vartheta^j + {}^{(L)}g_{ij}(x, y) {}^\vee \vartheta^i \otimes {}^\vee \vartheta^j \quad (9)$$

constructed as a Sasaki type lift from M .

One holds⁶⁻⁸ the following

Theorem 1 *The space $(\widetilde{TM}, \mathbf{F}, {}^{(L)}\mathbf{g})$ with almost complex form \mathbf{F} (8) defined by ${}^{(L)}N_j^i$, see (7), and canonical metric structure ${}^{(L)}\mathbf{g}$ (9) is an almost Kähler space with almost symplectic structure*

$$\begin{aligned} {}^{(L)}\theta &= {}^{(L)}\theta_{\alpha\beta}(x, y) \vartheta^\alpha \wedge \vartheta^\beta \\ &= {}^{(L)}g_{ij}(x, y) {}^\vee \vartheta^i \wedge \vartheta^j. \end{aligned} \quad (10)$$

Proof. It is evident if we define

$${}^{(L)}\theta(\mathbf{X}, \mathbf{Y}) \doteq {}^{(L)}\mathbf{g}(\mathbf{F}\mathbf{X}, \mathbf{Y})$$

and put $\mathbf{X} = \mathbf{e}_\alpha$ and $\mathbf{Y} = \mathbf{e}_\beta$. ■

^eOn the tangent bundle the indices related to the base space run the same values as those related to fibers: we can use the same symbols but have to distinguish like ${}^\vee e_k$ certain irreducible v-components with respect to, (or for) N-adapted bases and cobases.

We conclude that any regular Lagrange mechanics can be geometrized as an almost Kähler space with N-connection distribution. In a such Lagrange-Kähler nonholonomic manifold, the fundamental geometric structures (semispray, N-connection, almost complex structure and canonical metric on \widetilde{TM}) are defined by the fundamental Lagrange function $L(x, y)$.

Remark 1 *For applications in optics of nonhomogeneous media and gravity (see, for instance, references^{7,9,11,12}) one considers metric forms of type $g_{ij} \sim e^{\lambda(x,y)} {}^{(L)}g_{ij}(x, y)$ which can not be derived from a mechanical Lagrangian. In the so-called generalized Lagrange geometry one considers Sasaki type metrics (9) with certain general coefficients both for the metric and N-connection, i.e., when ${}^{(L)}g_{ij} \rightarrow g_{ij}(x, y)$, and ${}^{(L)}N_j^i \rightarrow N_j^i(x, y)$.^f*

Remark 2 *Finsler geometry with the fundamental Finsler function $F(x, y)$, being homogeneous of type $F(x, \lambda y) = \lambda F(x, y)$, for nonzero $\lambda \in \mathbb{R}$, may be considered as a particular case of Lagrange geometry when $L = F^2$.^g We shall apply the methods of Finsler geometry and its almost Kähler models in this work. Nevertheless, because the generalized Lagrange spaces are very general ones enabled with N-anholonomic structure inducing a corresponding almost symplectic structure we shall emphasize just such geometric configurations.*

Remark 3 *It is also proved that both generalized Lagrange and Finsler geometries can be modelled on Riemannian-Cartan N-anholonomic manifolds^{13,30-32} if off-diagonal metrics and N-connections are introduced into consideration.*

Now we shall demonstrate how N-anholonomic configurations can be defined in gravity theories. In this case, it is convenient to work on a general manifold \mathbf{V} , $\dim \mathbf{V} = n + m$ with global splitting, instead of the tangent bundle \widetilde{TM} .

C N-connections in gravity

Let us consider a metric structure on \mathbf{V} with the coefficients defined with respect to a local coordinate basis $du^\alpha = (dx^i, dy^a)$,

$$\mathbf{g} = g_{\alpha\beta}(u) du^\alpha \otimes du^\beta$$

^fIn this case, we can similarly define an almost Kähler N-anholonomic space $(\widetilde{TM}, \mathbf{F}, \theta)$ with the geometric structures induced naturally by the N-connection.

^gIn another turn, there is a proof⁵⁹ that any Lagrange fundamental function L can be modelled as a singular case in a certain class Finsler geometries of extra dimension.

with

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \quad (11)$$

A metric, for instance, parametrized in the form (11) is generic off-diagonal if it can not be diagonalized by any coordinate transforms. Performing a frame transform with the coefficients

$$\mathbf{e}_\alpha^{\underline{a}}(u) = \begin{bmatrix} e_i^{\underline{a}}(u) & N_i^b(u) e_b^{\underline{a}}(u) \\ 0 & e_a^{\underline{a}}(u) \end{bmatrix}, \quad (12)$$

$$\mathbf{e}_{\underline{\beta}}^{\beta}(u) = \begin{bmatrix} e_i^{\beta}(u) & -N_k^b(u) e_{\underline{i}}^k(u) \\ 0 & e_{\underline{a}}^{\beta}(u) \end{bmatrix}, \quad (13)$$

we write equivalently the metric in the form

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}(u) \vartheta^\alpha \otimes \vartheta^\beta = g_{ij}(u) \vartheta^i \otimes \vartheta^j + h_{ab}(u) \vartheta^a \otimes \vartheta^b, \quad (14)$$

where $g_{ij} \doteq \mathbf{g}(e_i, e_j)$ and $h_{ab} \doteq \mathbf{g}(e_a, e_b)$ and

$$\mathbf{e}_\alpha = \mathbf{e}_\alpha^{\underline{a}} \partial_{\underline{a}} \text{ and } \vartheta^\beta = \mathbf{e}_{\underline{\beta}}^{\beta} du^{\underline{\beta}}.$$

are vielbeins of type (2) and (3) defined for arbitrary $N_i^b(u)$. We can consider a special class of manifolds provided with a global splitting into conventional “horizontal” and “vertical” subspaces (1) induced by the “off-diagonal” terms $N_i^b(u)$ and prescribed type of nonholonomic frame structure.

If the manifold \mathbf{V} is (pseudo) Riemannian, there is a unique linear connection (the Levi-Civita connection) ∇ , which is metric, $\nabla \mathbf{g} = \mathbf{0}$, and torsionless, $\nabla T = 0$. Nevertheless, the connection ∇ is not adapted to the nonintegrable distribution induced by $N_i^b(u)$. In this case,^h it is more convenient to work with more general classes of linear connections which are N-adapted but contain nontrivial torsion coefficients because of nontrivial nonholonomy coefficients $W_{\alpha\beta}^\gamma$ (5).

For a splitting of a (pseudo) Riemannian–Cartan space of dimension $(n + m)$ (under certain constraints, we can consider (pseudo) Riemannian configurations), the Lagrange and Finsler type geometries were modelled by N-anholonomic structures as exact solutions of gravitational field equations^{9–13,31,32}. In this paper, we shall concentrate on N-anholonomic almost complex structures of vacuum gravity which can be naturally defined as $(n + n)$ configurations, in general, embedded in certain spaces of dimension $(n + m)$, $m \geq n$.

^hFor instance, in order to construct exact solutions parametrized by generic off-diagonal metrics, or for investigating nonholonomic frame structures in gravity models with non-trivial torsion.

III Connections on Almost Symplectic N-anholonomic Manifolds

The geometric constructions can be adapted to the N-connection structure:

Definition 4 *A distinguished connection (d-connection) \mathbf{D} on a manifold \mathbf{V} is a linear connection conserving under parallelism the Whitney sum (1) defining a general N-connection. Equivalently, $\mathbf{D}\mathbf{P} = 0$, \mathbf{P} being the almost product structure defined by the N-connection.*

The N-adapted components $\Gamma_{\beta\gamma}^\alpha$ of a d-connection $\mathbf{D}_\alpha = (\delta_\alpha]\mathbf{D})$ are defined by the equations

$$\mathbf{D}_\alpha \delta_\beta = \Gamma_{\alpha\beta}^\gamma \delta_\gamma,$$

or

$$\Gamma_{\alpha\beta}^\gamma(u) = (\mathbf{D}_\alpha \delta_\beta)]\delta^\gamma. \quad (15)$$

In its turn, this defines a N-adapted splitting into h- and v-covariant derivatives, $\mathbf{D} = D + {}^\vee D$, where $D_k = (L_{jk}^i, L_{bk}^a)$ and ${}^\vee D_c = (C_{jk}^i, C_{bc}^a)$ are introduced as corresponding h- and v-parametrizations of (15),

$$L_{jk}^i = (\mathbf{D}_k e_j)]\vartheta^i, \quad L_{bk}^a = (\mathbf{D}_k e_b)]\vartheta^a, \quad C_{jc}^i = (\mathbf{D}_c e_j)]\vartheta^i, \quad C_{bc}^a = (\mathbf{D}_c e_b)]\vartheta^a.$$

The components $\Gamma_{\alpha\beta}^\gamma = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a)$ completely define a d-connection \mathbf{D} on a N-anholonomic manifold \mathbf{V} .

The simplest way to perform computations with d-connections is to use N-adapted differential forms like $\Gamma_\beta^\alpha = \Gamma_{\beta\gamma}^\alpha \vartheta^\gamma$ with the coefficients defined with respect to (3) and (2).

We shall say that a d-connection \mathbf{D} preserves an almost symplectic 2-form, of Lagrange type ${}^{(L)}\theta$ (10) (or any general one, θ) defined from a generalized Lagrange geometry or N-anholonomic gravity model, if

$$\mathbf{D}\theta = 0 \quad (16)$$

or

$$\mathbf{Z}(\theta(\mathbf{X}, \mathbf{Y})) = \theta(\mathbf{D}_Z \mathbf{X}, \mathbf{Y}) + \theta(\mathbf{X}, \mathbf{D}_Z \mathbf{Y})$$

for any d-vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in T\mathbf{V}$.

Theorem 2 *The torsion $\mathcal{T}^\alpha \doteq \mathbf{D}\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta$ of a d-connection has the irreducible h- v- components (d-torsions) with N-adapted coefficients*

$$\begin{aligned} T_{jk}^i &= L_{[jk]}^i, \quad T_{ja}^i = -T_{aj}^i = C_{ja}^i, \quad T_{ji}^a = \Omega_{ji}^a, \\ T_{bi}^a &= T_{ib}^a = \frac{\partial N_i^a}{\partial y^b} - L_{bi}^a, \quad T_{bc}^a = C_{[bc]}^a, \end{aligned} \quad (17)$$

where $L^i_{[jk]} = L^i_{jk} - L^i_{kj}$ and so on.

Proof. By a straightforward calculation we can verify the formulas. ■

Remark 4 *The Levi-Civita linear connection $\nabla = \{\nabla \Gamma^\alpha_{\beta\gamma}\}$, with vanishing both torsion and nonmetricity, is not adapted to the global splitting (1). In fact, if ∇ was adapted, then $\nabla \mathbf{P} = 0$, \mathbf{P} being the almost product structure defined by the N -connection, and then, as ∇ is torsionless, one obtains by means of the Lemma 2.1.6 of⁶⁰ that the Nijenhuis tensor field $N_{\mathbf{P}}$ vanishes, thus proving that both vertical and horizontal distributions are involutive in the sense of Frobenius theorem, which is not our case of anholonomic manifolds. Then, we must look for another connection to study the geometry of these manifolds.*

One holds:

Proposition 3 *There is a preferred, canonical d -connection structure, $\hat{\mathbf{D}}$, on N -anholonomic manifold \mathbf{V} constructed only from the metric and N -connection coefficients $[g_{ij}, h_{ab}, N^a_i]$ and satisfying the conditions $\hat{\mathbf{D}}\mathbf{g} = 0$ and $\hat{T}^i_{jk} = 0$ and $\hat{T}^a_{bc} = 0$.*

Proof. By straightforward calculations with respect to the N -adapted bases (3) and (2), we can verify that the connection

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \nabla \Gamma^\alpha_{\beta\gamma} + \hat{\mathbf{P}}^\alpha_{\beta\gamma} \quad (18)$$

with the deformation d -tensorⁱ

$$\hat{\mathbf{P}}^\alpha_{\beta\gamma} = (P^i_{jk} = 0, P^a_{bk} = \frac{\partial N^a_k}{\partial y^b}, P^i_{jc} = -\frac{1}{2}g^{ik}\Omega^a_{kj}h_{ca}, P^a_{bc} = 0)$$

satisfies the conditions of this Proposition. It should be noted that, in general, the components \hat{T}^i_{ja} , \hat{T}^a_{ji} and \hat{T}^a_{bi} are not zero. This is an anholonomic frame (or, equivalently, off-diagonal metric) effect. ■

ⁱ $\hat{\mathbf{P}}^\alpha_{\beta\gamma}$ is a tensor field of type (1,2). As is well known, the sum of a linear connection and a tensor field of type (1,2) is a new linear connection.

With respect to the N-adapted frames, the coefficients $\widehat{\Gamma}^\gamma_{\alpha\beta} = \left(\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc}\right)$ are computed:

$$\begin{aligned}\widehat{L}^i_{jk} &= \frac{1}{2}g^{ir}\left(\frac{\delta g_{jr}}{\partial x^k} + \frac{\delta g_{kr}}{\partial x^j} - \frac{\delta g_{jk}}{\partial x^r}\right), \\ \widehat{L}^a_{bk} &= \frac{\partial N^a_k}{\partial y^b} + \frac{1}{2}h^{ac}\left(\frac{\delta h_{bc}}{\partial x^k} - \frac{\partial N^d_k}{\partial y^b}h_{dc} - \frac{\partial N^d_k}{\partial y^c}h_{db}\right), \\ \widehat{C}^i_{jc} &= \frac{1}{2}g^{ik}\frac{\partial g_{jk}}{\partial y^c}, \\ \widehat{C}^a_{bc} &= \frac{1}{2}h^{ad}\left(\frac{\partial h_{bd}}{\partial y^c} + \frac{\partial h_{cd}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d}\right).\end{aligned}\tag{19}$$

For the canonical d-connection there are satisfied the conditions of vanishing of torsion on the h-subspace and v-subspace, i.e., $\widehat{T}^i_{jk} = \widehat{T}^a_{bc} = 0$. In more general cases, such components of torsion are not zero, for instance, the metric d-connections of type $\Gamma^\gamma_{\alpha\beta} = \left(\widehat{L}^i_{jk} + l^i_{jk}(u), \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc} + c^a_{bc}(u)\right)$ is also compatible with metric (14) and has nontrivial T^i_{jk} and \widehat{T}^a_{bc} .

Let us consider a special case with $\dim \mathbf{V} = n+n$, $h_{ab} \rightarrow g_{ij}$ and $N^a_i \rightarrow N^j_i$ in (14) when a tangent bundle structure is locally modelled on \mathbf{V} . We denote a such space by $\widetilde{\mathbf{V}}_{(n,n)}$. One holds:

Theorem 4 *The canonical d-connection $\widehat{\mathbf{D}}$ (19) for a local modelling of a \widetilde{TM} space on $\widetilde{\mathbf{V}}_{(n,n)}$ is defined by $\widehat{\Gamma}^\gamma_{\alpha\beta} = (\widehat{L}^i_{jk}, \widehat{C}^i_{jk})$ with*

$$\widehat{L}^i_{jk} = \frac{1}{2}g^{ir}\left(\frac{\delta g_{jr}}{\partial x^k} + \frac{\delta g_{kr}}{\partial x^j} - \frac{\delta g_{jk}}{\partial x^r}\right), \widehat{C}^i_{jk} = \frac{1}{2}g^{ir}\left(\frac{\partial g_{jr}}{\partial x^k} + \frac{\partial g_{kr}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^r}\right).\tag{20}$$

This d-connection is almost Hermitian, i.e., it is compatible with the almost Hermitian structure (\mathbf{g}, \mathbf{F}) , when

$$\widehat{\mathbf{D}}\theta = 0 \text{ and } \widehat{\mathbf{D}}\mathbf{F} = 0\tag{21}$$

for a 2-form ^j

$$\theta = \theta_{\alpha\beta}(x, y)\vartheta^\alpha \wedge \vartheta^\beta = g_{ij}(x, y) \vartheta^i \wedge \vartheta^j.$$

Proof. It is similar to that for the Theorem 1. ■

^jIn an intrinsic way, $\theta(X, Y) = \mathbf{g}(\mathbf{F}X, Y)$.

On almost symplectic manifolds, usually there are considered symmetric linear connections. In our case, we can always define a symmetric d-connection by taking the symmetric part ^k of $\Gamma_{\alpha\beta}^\gamma$,

$$\mathbf{S}_{\alpha\beta}^\gamma = \frac{1}{2} (\Gamma_{\alpha\beta}^\gamma + \Gamma_{\beta\alpha}^\gamma), \quad (22)$$

where $\Gamma_{\alpha\beta}^\gamma = (\widehat{L}_{jk}^i + l_{jk}^i(u), \widehat{C}_{jk}^i + c_{jk}^i(u))$. On a N-anholonomic manifold $\widetilde{\mathbf{V}}_{(n,n)}$, an almost symplectic form θ is not closed, i. e. $d\theta \neq \mathbf{0}$. But it may be closed under the action of N-adapted derivatives (2) and differentials (3) when

$$\delta\theta = \delta(\theta_{\alpha\beta}(x, y) \vartheta^\alpha \wedge \vartheta^\beta) = 0,$$

which means that

$$\mathbf{e}_\gamma \theta_{\alpha\beta} + \mathbf{e}_\alpha \theta_{\gamma\beta} + \mathbf{e}_\beta \theta_{\alpha\gamma} = 0. \quad (23)$$

The condition (16) written in N-adapted bases results in

$$\mathbf{e}_\gamma \theta_{\alpha\beta} = \Gamma_{\alpha\gamma\beta} - \Gamma_{\beta\gamma\alpha}$$

for $\Gamma_{\alpha\gamma\beta} \doteq \theta_{\alpha\tau} \Gamma_{\gamma\beta}^\tau$.

Definition 5 *An almost symplectic 2-form θ is N-symplectic if it satisfies the conditions (23).*

There is a relation between the set of all d-connections \mathbf{D} for which $\mathbf{D}\theta = \mathbf{0}$ for any given θ and \mathbf{N} and the set of all symmetric connections on $\widetilde{\mathbf{V}}_{(n,n)}$. By straightforward calculations we can verify that

$$\Gamma_{\alpha\gamma\beta} = \frac{1}{2} (\mathbf{e}_\alpha \theta_{\gamma\beta} - \mathbf{e}_\gamma \theta_{\alpha\beta} - \mathbf{e}_\beta \theta_{\alpha\gamma}) + (\mathbf{S}_{\alpha\gamma\beta} - \mathbf{S}_{\gamma\beta\alpha} + \mathbf{S}_{\beta\gamma\alpha}) \quad (24)$$

is inverse to (22), which for almost symplectic $\theta_{\alpha\beta}$ satisfying the conditions (23) simplifies to

$$\Gamma_{\alpha\gamma\beta} = \mathbf{e}_\alpha \theta_{\gamma\beta} + (\mathbf{S}_{\alpha\gamma\beta} - \mathbf{S}_{\gamma\beta\alpha} + \mathbf{S}_{\beta\gamma\alpha}).$$

On holonomic manifolds with trivial N-connection, the formulas (23) and (24) transform into those from reference³ with $\mathbf{e}_\alpha \rightarrow \partial/\partial u^\alpha$. We may conclude that N-anholonomic transforms map symplectic forms in almost symplectic ones but preserve the main symmetry properties and compatibility with the linear connection structure if the computations are performed with respect to N-adapted bases.

^kIn coordinate-free notation, $\mathbf{S}_X Y = \frac{1}{2} (\mathbf{D}_X Y + \mathbf{D}_Y X + [X, Y])$.

IV Curvature of N-symplectic d-Connections

Let \mathbf{V} (or $\tilde{\mathbf{V}}_{(n,n)}$) be an N-anholonomic manifold provided with a metric d-connection $\mathbf{\Gamma}_\gamma^\alpha$.

Definition 6 A Fedosov N-anholonomic manifold is defined by an almost symplectic d-connection and almost complex structure induced by the N-connection.

Definition 7 A Lagrange-Fedosov manifold is a Fedosov N-anholonomic manifold with the N-connection and almost complex structure defined by the fundamental Lagrange function, see Theorem 1.

The curvature of a symplectic d-connection \mathbf{D} is defined by the usual formula

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} \doteq \mathbf{D}_X \mathbf{D}_Y \mathbf{Z} - \mathbf{D}_Y \mathbf{D}_X \mathbf{Z} - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]} \mathbf{Z}.$$

Because on N-anholonomic spaces the “simplest” adapted to the N-connection induced almost complex structures is defined by the canonical d-connection, it is convenient to use it as a symplectic d-connection.

By straightforward calculations we prove:

Theorem 5 The curvature $\mathcal{R}_\beta^\alpha \doteq \mathbf{D}\mathbf{\Gamma}_\beta^\alpha - \mathbf{\Gamma}_\beta^\gamma \wedge \mathbf{\Gamma}_\gamma^\alpha$ of a d-connection $\mathbf{\Gamma}_\gamma^\alpha$ has the irreducible h- v- components (d-curvatures) of $\mathbf{R}_{\beta\gamma\delta}^\alpha$,

$$\begin{aligned} R_{hjk}^i &= e_k L_{hj}^i - e_j L_{hk}^i + L_{hj}^m L_{mk}^i - L_{hk}^m L_{mj}^i - C_{ha}^i \Omega_{kj}^a, \\ R_{bjk}^a &= e_k L_{bj}^a - e_j L_{bk}^a + L_{bj}^c L_{ck}^a - L_{bk}^c L_{cj}^a - C_{bc}^a \Omega_{kj}^c, \\ R_{jka}^i &= e_a L_{jk}^i - D_k C_{ja}^i + C_{jb}^i T_{ka}^b, \\ R_{bka}^c &= e_a L_{bk}^c - D_k C_{ba}^c + C_{bd}^c T_{ka}^d, \\ R_{jbc}^i &= e_c C_{jb}^i - e_b C_{jc}^i + C_{jb}^h C_{hc}^i - C_{jc}^h C_{hb}^i, \\ R_{bcd}^a &= e_d C_{bc}^a - e_c C_{bd}^a + C_{bc}^e C_{ed}^a - C_{bd}^e C_{ec}^a. \end{aligned} \tag{25}$$

Remark 5 For an N-anholonomic manifold $\tilde{\mathbf{V}}_{(n,n)}$ provided with N-symplectic canonical d-connection $\hat{\mathbf{\Gamma}}_{\gamma\alpha\beta} = \theta_{\gamma\tau} \hat{\mathbf{\Gamma}}_{\alpha\beta}^\tau$, see (20), the d-curvatures (25) reduces to three irreducible components

$$\begin{aligned} R_{hjk}^i &= e_k L_{hj}^i - e_j L_{hk}^i + L_{hj}^m L_{mk}^i - L_{hk}^m L_{mj}^i - C_{ha}^i \Omega_{kj}^a, \\ R_{jka}^i &= e_a L_{jk}^i - D_k C_{ja}^i + C_{jb}^i T_{ka}^b, \\ R_{bcd}^a &= e_d C_{bc}^a - e_c C_{bd}^a + C_{bc}^e C_{ed}^a - C_{bd}^e C_{ec}^a \end{aligned} \tag{26}$$

where all indices $i, j, k \dots$ and a, b, \dots run the same values but label the components with respect to different h- or v-frames.

The indices of the components of the curvature tensor are lowered as

$$\mathbf{R}_{\tau\beta\gamma\delta} = \theta_{\tau\alpha} \mathbf{R}^{\alpha}_{\beta\gamma\delta}.$$

For Lagrange–Fedosov manifolds, the 2-form $\theta_{\tau\alpha}$ has the coefficients defined by the metric structure and Lagrangian, see (10). In this case we can apply the canonical d-connection and the d-metric for definition of the curvature of symplectic d-connections.

Contracting respectively the components of (25) and (26) we prove:

Corollary 6 *The Ricci d-tensor $\mathbf{R}_{\alpha\beta} \doteq \mathbf{R}^{\tau}_{\alpha\beta\tau}$ has the irreducible h- v- components*

$$R_{ij} \doteq R^k_{ijk}, \quad R_{ia} \doteq -R^k_{ika}, \quad R_{ai} \doteq R^b_{aib}, \quad R_{ab} \doteq R^c_{abc}, \quad (27)$$

for a general N-holonomic manifold \mathbf{V} , and

$$R_{ij} \doteq R^k_{ijk}, \quad R_{ia} \doteq -R^k_{ika}, \quad R_{ab} \doteq R^c_{abc}, \quad (28)$$

for an N-anholonomic manifold $\tilde{\mathbf{V}}_{(n,n)}$.

Corollary 7 *The scalar curvature of a d-connection is*

$$\begin{aligned} \overleftarrow{\mathbf{R}} &\doteq \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab}, \text{ for } \mathbf{V}; \\ &= 2g^{ij} R_{ij}, \text{ for } \tilde{\mathbf{V}}_{(n,n)}. \end{aligned}$$

Corollary 8 *The Einstein d-tensor is computed $\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} \overleftarrow{\mathbf{R}}$.*

In modern gravity theories, one considers more general linear connections generated by deformations of type $\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} + \mathbf{P}^{\alpha}_{\beta\gamma}$. We can split all geometric objects into canonical and post-canonical pieces which results in N-adapted geometric constructions. For instance,

$$\mathcal{R}^{\alpha}_{\beta} = \widehat{\mathcal{R}}^{\alpha}_{\beta} + \mathbf{D}\mathcal{P}^{\alpha}_{\beta} + \mathcal{P}^{\alpha}_{\gamma} \wedge \mathcal{P}^{\gamma}_{\beta} \quad (29)$$

for $\mathcal{P}^{\alpha}_{\beta} = \mathbf{P}^{\alpha}_{\beta\gamma} \vartheta^{\gamma}$. This way, for almost complex geometries, the d-tensors (26) and (28) can be redefined just for symmetrized d-connections compatible with the almost complex structure.

V Einstein Flat N–Anholonomic Manifolds

In terms of differential forms, the vacuum Einstein equations are written

$$\eta_{\alpha\beta\gamma} \wedge \widehat{\mathcal{R}}^{\beta\gamma} = 0, \quad (30)$$

where, for the volume 4–form $\eta \doteq *1$ with the Hodge operator “*”, $\eta_\alpha \doteq \mathbf{e}_\alpha \lrcorner \eta$, $\eta_{\alpha\beta} \doteq \mathbf{e}_\beta \lrcorner \eta_\alpha$, $\eta_{\alpha\beta\gamma} \doteq \mathbf{e}_\gamma \lrcorner \eta_{\alpha\beta}$, ... and $\widehat{\mathcal{R}}^{\beta\gamma}$ is the curvature 2–form. The deformation of connection (18) defines a deformation of the curvature tensor of type (29) but with respect to the curvature of the Levi–Civita connection, $\nabla \mathcal{R}^{\beta\gamma}$. The gravitational field equations (30) transforms into

$$\eta_{\alpha\beta\gamma} \wedge \nabla \mathcal{R}^{\beta\gamma} + \eta_{\alpha\beta\gamma} \wedge \nabla \mathcal{Z}^{\beta\gamma} = 0, \quad (31)$$

where $\nabla \mathcal{Z}^{\beta\gamma} = \nabla \mathcal{P}^{\beta\gamma} + \mathcal{P}^{\beta\gamma} \wedge \mathcal{P}^\alpha{}_\alpha$.

A subclass of solutions of the gravitational field equations for the canonical d–connection defines also solutions of the Einstein equations for the Levi–Civita connection if and only if

$$\eta_{\alpha\beta\gamma} \wedge \nabla \mathcal{Z}^{\beta\gamma} = 0. \quad (32)$$

This property is very important for constructing exact solutions in Einstein and string gravity, parametrized by generic off–diagonal metrics and anholonomic frames with associated N–connection structure (see reviews of results in references^{30,31} and³²).

A The ansatz for metric

In this subsection we investigate a class of five dimensional vacuum Einstein solutions with nontrivial associated N–connection and generic off–diagonal metric. We analyze the conditions when such solutions reduce to four dimensions and posses almost complex structure.

Let us consider a five dimensional ansatz for the metric (14) and frame (3) when $u^\alpha = (x^i, y^4 = v, y^5)$; $i = 1, 2, 3$ and the coefficients

$$\begin{aligned} g_{ij} &= \text{diag}[g_1 = \pm \varpi(x^k, v), \varpi(x^k, v)g_2(x^2, x^3), \varpi(x^k, v)g_3(x^2, x^3)], \\ h_{ab} &= \text{diag}[\varpi(x^k, v)h_4(x^k, v), \varpi(x^k, v)h_5(x^k, v)], \\ N_i^4 &= w_i(x^k, v), N_i^5 = n_i(x^k, v) \end{aligned} \quad (33)$$

are some functions of necessary smooth class. The partial derivative are briefly denoted $a^\bullet = \partial a / \partial x^2$, $a' = \partial a / \partial x^3$, $a^* = \partial a / \partial v$.

Theorem 9 *The vacuum Einstein equations (30) for the canonical d -connection (18) constructed from data (33) are equivalent to the system of equations*

$$g_3^{\bullet\bullet} - \frac{g_2^{\bullet} g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2'' - \frac{g_2' g_3'}{2g_3} - \frac{(g_2')^2}{2g_2} = 0, \quad (34)$$

$$h_5^{**} - h_5^* (\ln \left| \sqrt{|h_4 h_5|} \right|)^* = 0, \quad (35)$$

$$w_i \beta + \alpha_i = 0, \quad (36)$$

$$n_i^{**} + n_i^* = 0, \quad (37)$$

where

$$\begin{aligned} \alpha_i &= \partial_i h_5^* - h_5^* \partial_i \ln \left| \sqrt{|h_4 h_5|} \right|, \quad \beta = h_5^{**} - h_5^* \left[\ln \left| \sqrt{|h_4 h_5|} \right| \right]^*, \\ \gamma &= 3h_5^*/2h_5 - h_4^*/h_4 \end{aligned} \quad (38)$$

$h_4^* \neq 0$ and $h_5^* \neq 0$ and the functions h_4 and ϖ must satisfy certain additional conditions

$$\widehat{\delta}_i h_4 = 0 \text{ and } \widehat{\delta}_i \varpi = 0, \quad (39)$$

for any $\zeta_i(x^k, v)$ defining $\widehat{\delta}_i = \partial_i - (w_i + \zeta_i)\partial_4 + n_i\partial_5$.

Proof. It is a straightforward calculation, see similar ones in^{31,9,11}. ■

We note that the conditions (39) are satisfied if

$$\varpi^{q_1/q_2} = h_4 \quad (40)$$

for some nonzero integers q_1 and q_2 and ζ_i defined from the equations

$$\partial_i \varpi - (w_i + \zeta_i) \varpi^* = 0. \quad (41)$$

Remark 6 *Under the conditions of the Theorem 9, we can also consider d -metrics with $h_5^* = 0$ for such functions $h_4 = h^\#(x^i, v)$ when*

$$\lim_{h_5^* \rightarrow 0} \left\{ h_5^* [\ln \left| \sqrt{|h^\# h_5|} \right|]^* \right\} \rightarrow 0$$

and

$$\lim_{h_5^* \rightarrow 0} \left\{ h_5^* \partial_i \ln \left| \sqrt{|h^\# h_5|} \right| \right\} \rightarrow 0.$$

In this cases, the equations (35) and (36) will be satisfied by any $h^\#(x^i, v)$ and $w_i(x^i, v)$ and we may take $n_i^* = n_{[1]i}(x^i) h^\#(x^i, v)$ in order to satisfy (37).

Theorem 10 *The system of gravitational field equations (30) for the ansatz (33) can be solved in general form if there are given certain values of functions $g_2(x^2, x^3)$ (or, inversely, $g_3(x^2, x^3)$), $h_4(x^i, v)$ (or, inversely, $h_5(x^i, v)$).*

Proof. We outline the main steps of constructing exact solutions proving this Theorem, see detailed computations presented in the Proof of Theorem 4.3 from reference³¹.

- The general solution of equation (34) can be written in the form

$$\lambda = g_{[0]} \exp[a_2 \tilde{x}^2(x^2, x^3) + a_3 \tilde{x}^3(x^2, x^3)], \quad (42)$$

where $g_{[0]}$, a_2 and a_3 are some constants and the functions $\tilde{x}^{2,3}(x^2, x^3)$ define any coordinate transforms $x^{2,3} \rightarrow \tilde{x}^{2,3}$ for which the two dimensional line element becomes conformally flat, i.e.,

$$g_2(x^2, x^3)(dx^2)^2 + g_3(x^2, x^3)(dx^3)^2 \rightarrow \lambda(x^2, x^3) [(d\tilde{x}^2)^2 + \epsilon(d\tilde{x}^3)^2], \quad (43)$$

where $\epsilon = \pm 1$ for a corresponding signature. In coordinates $\tilde{x}^{2,3}$, the equation (34) transform into

$$\lambda(\lambda^{\bullet\bullet} + \lambda'') - \lambda^{\bullet} - \lambda' = 0$$

or

$$\ddot{\psi} + \psi'' = 0, \quad (44)$$

for $\psi = \ln |\lambda|$. There are three alternative possibilities to generate solutions of (34). For instance, we can prescribe that $g_2 = g_3$ and get the equation (44) for $\psi = \ln |g_2| = \ln |g_3|$. If we suppose that $g_2' = 0$, for a given $g_2(x^2)$, we obtain from (34)

$$g_3^{\bullet\bullet} - \frac{g_2^{\bullet} g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} = 0$$

which can be integrated exactly. Similarly, we can generate solutions for a prescribed $g_3(x^3)$ in the equation

$$g_2'' - \frac{g_2' g_3'}{2g_3} - \frac{(g_2')^2}{2g_2} = 0. \quad (45)$$

We note that a transform (43) is always possible for 2D metrics and the explicit form of solutions depends on chosen system of 2D coordinates and on the signature $\epsilon = \pm 1$. In the simplest case, the equation (34) is solved by arbitrary two functions $g_2(x^3)$ and $g_3(x^2)$.

- The equation (35) relates two functions $h_4(x^i, v)$ and $h_5(x^i, v)$ following two possibilities:

a) to compute

$$\begin{aligned}\sqrt{|h_5|} &= h_{5[1]}(x^i) + h_{5[2]}(x^i) \int \sqrt{|h_4(x^i, v)|} dv, \quad h_4^*(x^i, v) \neq 0; \\ &= h_{5[1]}(x^i) + h_{5[2]}(x^i) v, \quad h_4^*(x^i, v) = 0,\end{aligned}\quad (46)$$

for some functions $h_{5[1,2]}(x^i)$ stated by boundary conditions;

b) or, inversely, to compute h_4 for a given $h_5(x^i, v)$, $h_5^* \neq 0$,

$$\sqrt{|h_4|} = h_{[0]}(x^i) (\sqrt{|h_5(x^i, v)|})^*, \quad (47)$$

with $h_{[0]}(x^i)$ given by boundary conditions.

- The exact solutions of (36) for $\beta \neq 0$ are defined from an algebraic equation, $w_i \beta + \alpha_i = 0$, where the coefficients β and α_i are computed as in formulas (38) by using the solutions for (34) and (35). The general solution is

$$w_k = \partial_k \ln[\sqrt{|h_4 h_5|}/|h_5^*|]/\partial_v \ln[\sqrt{|h_4 h_5|}/|h_5^*|], \quad (48)$$

with $\partial_v = \partial/\partial v$ and $h_5^* \neq 0$. If $h_5^* = 0$, or even $h_5^* \neq 0$ but $\beta = 0$, the coefficients w_k could be arbitrary functions on (x^i, v) . For the vacuum Einstein equations this is a degenerated case imposing the compatibility conditions $\beta = \alpha_i = 0$, which are satisfied, for instance, if the h_4 and h_5 are related as in the formula (47) but with $h_{[0]}(x^i) = \text{const.}$

- Having defined h_4 and h_5 and computed γ from (38) we can solve the equation (37) by integrating on variable “ v ” the equation $n_i^{**} + \gamma n_i^* = 0$. The exact solution is

$$\begin{aligned}n_k &= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [h_4/(\sqrt{|h_5|})^3] dv, \quad h_5^* \neq 0; \\ &= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int h_4 dv, \quad h_5^* = 0; \\ &= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [1/(\sqrt{|h_5|})^3] dv, \quad h_4^* = 0,\end{aligned}\quad (49)$$

for some functions $n_{k[1,2]}(x^i)$ stated by boundary conditions.

The exact solution of (41) is given by some functions $\zeta_i = \zeta_i(x^i, v)$ if both $\partial_i \varpi = 0$ and $\varpi^* = 0$, we chose $\zeta_i = 0$ for $\varpi = \text{const}$, and

$$\begin{aligned}\zeta_i &= -w_i + (\varpi^*)^{-1} \partial_i \varpi, \quad \varpi^* \neq 0, \\ &= (\varpi^*)^{-1} \partial_i \varpi, \quad \varpi^* \neq 0, \text{ for vacuum solutions.}\end{aligned}\tag{50}$$

■

The Theorem 10 states a general method of constructing five dimensional exact solutions in various gravity models with generic off-diagonal metrics, nonholonomic frames and, in general, with nontrivial torsion. Such solutions are with associated N-connection structure. This method can be also applied in order to generate, for instance, certain Finsler or Lagrange configurations as v-irreducible components, or for a certain class of conformal factors $\varpi(x^i, v)$ for both h- and v-irreducible components. The five dimensional ansatz can not be used to generate directly standard Finsler or Lagrange geometries because the dimension of such spaces can not be an odd number. Nevertheless, the anholonomic frame method can be applied in order to generate four dimensional exact solutions containing Finsler-Lagrange configurations. For instance, a four dimensional configuration can be defined just by an ansatz (14) with the data (33) where the coefficients do not depend on coordinate x^1 and the metric is stated to be four dimensional with the conformal factor $\varpi(x^2, x^3, v)$.

B An example of induced almost Kähler gravity

Let us consider a four dimensional ansatz which may mimic under certain constraints a generalized Lagrange geometry and induced almost Kähler structure in Riemann-Cartan space:

$$\begin{aligned}\mathbf{g} &= \varpi(x^2, x^3, v) [g_{22}(x^2, x^3) dx^2 \otimes dx^2 + g_{33}(x^2, x^3) dx^3 \otimes dx^3 \\ &\quad + h_{44}(x^2, x^3, v) \delta y^4 \otimes \delta y^4 + h_{55}(x^2, x^3, v) \delta y^5 \otimes \delta y^5],\end{aligned}$$

where

$$\begin{aligned}\delta y^4 &= dv + w_2(x^2, x^3, v) dx^2 + w_3(x^2, x^3, v) dx^3, \\ \delta y^5 &= dy^5 + n_2(x^2, x^3, v) dx^2 + n_3(x^2, x^3, v) dx^3.\end{aligned}$$

This d-metric will define a class of vacuum solutions of the Einstein equations if the coefficients are subjected to the conditions of the Theorem 10, when the dependence on coordinate x^1 is eliminated. We put $g_{22} = g(x^3)$ and $g_{33} = 0$ to be a solution of (34) in the form (45), i.e.,

$$2gg'' - (g')^2 = 0$$

and choose $h_5 = 0$ and

$$h_4 = h^\#(x^3, v) = \frac{a^2}{|g(x^3) \times v|} g(x^3)$$

for $a = \text{const}$, which satisfies (35), see Remark 6. Taking any functions $w_{2,3}(x^2, x^3, v)$ and $n_{2,3}(x^2, x^3, v)$ satisfying

$$n_{2,3}^* = n_{2,3[0]}(x^2, x^3) h^\#(x^3, v)$$

we solve respectively the equations (36) and (37). We may take

$$\varpi = \varpi^\#(x^3, v) = [h^\#(x^3, v)]^{q_2/q_1}$$

like for (50). All such functions define a vacuum Einstein d-metric

$$\mathbf{g} = \varpi^\#(x^3, v) [g(x^3) dx^2 \otimes dx^2 + \frac{a^2}{|g(x^3) \times v^2|} g(x^3) \delta y^4 \otimes \delta y^4],$$

modelling an embedded generalized Lagrange geometry (it is a particular case of d-metrics considered in⁸, see formula (6.3), which in our case is derived from a gravity model). We construct a conformal almost Kähler geometry if we consider

$$\theta = \varpi^\#(x^3, v) g(x^3) \frac{a}{\sqrt{|g(x^3) \times v^2|}} \delta y^4 \wedge dx^2$$

and

$$\begin{aligned} \mathbf{F} = & \frac{\sqrt{|g(x^3) \times v^2|}}{a} \left(\frac{\partial}{\partial v} \otimes dx^2 + \frac{\partial}{\partial y^5} \otimes dx^3 \right) \\ & - \frac{a}{\sqrt{|g(x^3) \times v^2|}} \left(\frac{\delta}{\partial x^2} \otimes dv + \frac{\delta}{\partial x^3} \otimes dy^5 \right). \end{aligned}$$

Finally, we note that if we choose the functions $w_{2,3}(x^2, x^3, v)$ and $n_{2,3}(x^2, x^3, v)$ to parametrize a noncommutative structure, this vacuum gravitational space will possess a noncommutative symmetry like in^{31,32}. An alternative class of solutions can be generated if we put certain boundary conditions (for instance, for $v = t$ treated as a timelike coordinate, and one of the space coordinates x^2, x^3, y^5 running to infinite) when the N-connection coefficients possess a Lie algebra symmetry. In this case, we generate an explicit example of vacuum gravitational fields (in general, with nontrivial torsion) possessing Lie symmetries⁶¹. We can select such values of $w_{2,3}$ and $n_{2,3}$ when the conditions (32) are satisfied and the solutions coincide with those for the Levi Civita connection, but this is a very restricted case of N-connection geometry and associated almost complex structures.

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